§6. Quadratic ideals

One-sided ideals play less of a role in the alternative theory than in the associative theory since they are harder to come by. For one thing, we will see that there are some simple algebras (Cayley algebras) which have no proper one-sided ideals yet are not division algebras, so a simple algebra cannot be built up from its minimal left ideals. For another, one-sided ideals are harder to generate. For instance, the left ideal generated by an element x need not just consist of all left multiples ax of x: Ax may not be a left ideal since it may not be closed under left multiplication, the elements b(ax) may not be expressible as cx since in general $b(ax) \neq (ba)x$.

More useful are the quadratic ideals defined in terms of the operators U_x. A quadratic ideal is a subspace B such that

$$U_b\, A = b\, A\, b \subset B \quad \text{for all } b \in B\ .$$

(In contrast to a left ideal $Ab \subset B$ or right ideal $bA \subset B$.) If B is a quadratic ideal in A it need not remain a quadratic ideal in the unital hull $\hat{A}$: we have $U_B\hat{A} = \Phi U_B 1 + U_B A$, so to be quadratic in $\hat{A}$ we need all $b^2 = U_b 1$ in B as well as $U_b A \subset B$. We say B is a strict quadratic ideal if it remains a quadratic ideal in $\hat{A}$, i.e. is a quadratic ideal in A and contains the squares b^2 of all its elements.

The important thing is that quadratic ideals are quite simple to construct; for example, Ax and xA are always quadratic ideals, even if they are not left or right ideals.

6.1 (Construction of Quadratic ideals) Any one-sided ideal is also a strict quadratic ideal. If B is a quadratic ideal then xB and Bx and xBx are strict quadratic ideals for any $x \in A$.

Proof. Any one-sided ideal is automatically a quadratic ideal: if B is (say) a left ideal and $b \in B$ then $bAb = (bA)b \subset Ab \subset B$.

Suppose now B is given as a quadratic ideal. Then Bx is strict quadratic since by the right fundamental formula $U_{bx}\hat{A} = R_x U_b L_x \hat{A} \subset R_x U_b A \subset R_x B = Bx$ (by definition $U_b A \subset B$ if B is quadratic). Similarly for xB. These already imply xBx is strict quadratic, or we can argue $U_{xbx}\hat{A} = U_x U_b U_x \hat{A} \subset U_x U_b A \subset U_x B$ by the middle fundamental formula. ∎

6.2 (Construction of Principal Quadratic Ideals). For any element $b \in A$ we have principal quadratic ideals bA and Ab and bAb. ∎

We call these the right, left, and (two sided) principal quadratic ideals generated by b. Principal quadratic ideals are always strict.

For any elements $b, c \in A$ we can apply the construction twice to get quadratic ideals (bA)c and b(Ac), b(cA), (Ab)c. Throwing in more elements, we can keep the process going indefinitely: {(bA)c}d, {e[(bA)c]}d, etc.

WARNING. In this world you never get something for nothing, and in order to generate quadratic ideals so easily we must pay a price: the sum of two quadratic ideals need not be a quadratic ideal. The trouble is the presence of cross terms in a quadratic expression: if B, C are quadratic then $U_{b+c}a = U_b a + U_c a + U_{b,c}a$ where $U_b a$ is in B, $U_c a$ is in C, but $U_{b,c}a$ is nowhere.

6.3 Intersection Example. Although sums and products of quadratic ideals need not be quadratic, the intersection of any collection of quadratic (or strict quadratic) ideals B_α is again quadratic (or strict): if $b \in B = \bigcap B_\alpha$ then $U_b A \subset U_{B_\alpha} A \subset B_\alpha$ for all α, so $U_b A \subset \bigcap B_\alpha = B$. ∎

6.4 Subideal Example. If C is an ideal in B and B is an ideal in A, then (even in the associative case) C need not be an ideal in A. However, it is at least a strict quadratic ideal in A: for $c \in C$ we have $U_c \hat{A} = c(\hat{A}c) \subset C(\hat{A}B) \subset CB$ $(B \triangleleft A)$ $\subset C$ $(C \triangleleft B)$. ∎

6.5 Annihilator Example. If S is any subset of A the left annihilator $Ann_L(S) = \{x \in A \mid xS = 0\}$ is a strict quadratic ideal by left Moufang: $(x\hat{a}x)S = x(\hat{a}(xS)) = 0$. Similarly the right annihilator $Ann_R(S) = \{x \in A \mid Sx = 0\}$ is quadratic. It is not in general true that the left annihilator is a left ideal or the right annihilator a right ideal. [If S happens to be a right ideal, we will see its left annihilator is a left ideal, and dually]. ∎

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The Inverse Theorem 4.2 (v) - (vi) shows that an element x of a unital algebra is invertible iff the principal quadratic ideal $U_x A$ it generates is all of A. This leads to the useful observation that a proper quadratic ideal cannot contain an invertible element, for as soon as B contains an invertible x it contains all of $U_x A = A$.

An alternative algebra is a division algebra if it is unital and all nonzero elements are invertible. In the associative case this is equivalent to the condition that A have no proper left ideals (or right ideals), but this is not true in the alternative case - we will see that

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6.4 Subideal Example. If C is an ideal in B and B is an ideal in A, then (even in the associative case) C need not be an ideal in A. However, it is at least a strict quadratic ideal in A: for $c \in C$ we have $U_c\hat{A} = c(\hat{A}c) \subset c(\hat{A}B) \subset cB$ $(B \lhd A)$ $\subset C$ $(C \lhd B)$. $\square$

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6.6 Associative Example. We want to show that in a nice associative algebra A all quadratic ideals B are principal: $B = xAy = xA \cap Ay$. This result will be important for the structure theory in Chapter VIII; it shows that quadratic ideals are not far from one-sided ideals.

6.7 (Associative Quadratic Ideal Theorem) Any quadratic ideal B in a semisimple Artinian associative algebra A is principal: $B = eAf$ for idempotents e,f.

Proof. By the Artin-Wedderburn theory (Part 1) we know A is regular and a direct sum $A = \boxplus A_i$ of simple Artinian ideals; by regularity any quadratic ideal will have the form $B = U_B A$ $= \boxplus U_B A_i = \boxplus B_i$ for $B_i = U_B A_i = B \cap A_i$ quadratic in A_i, so if $B_i = e_i A_i f_i$ we will have $B = eAf$ for $e = \Sigma e_i$, $f = \Sigma f_i$.

We therefore assume A itself is simple and Artinian. We recall that all elements of A are regular, and that A has a.c.c. on idempotents. Consequently we can choose a pair (e,f) of idempotents maximal with respect to the property $eAf \subset B$. Nonzero e,f always exist (if $B \neq 0$): if $b \neq 0$ in B then by regularity $b = bdb$ for some d, so $e = bd$ and $f = db$ are nonzero idempotents with $eAf = bdAdb \subset bAb \subset B$.

We claim $B = eAf$. That is, we claim $eB = Bf = B$, or equivalently $e'B = Bf' = 0$ for $e' = 1-e$, $f' = 1-f$. We will prove only the second relation $Bf' = 0$ (the first follows by a dual argument). Suppose on the contrary $b = cf' \neq 0$ for some $c \in B$. We first show $b \in B$: by regularity $b \in bAb$, and $AeA = A$ since A is simple and $e \neq 0$, so $b \in b\,AeAb = cf'AeAc(1-f) \subset cAc - cf'AeAf$ where $cAc = U_c A \subset B$ and $cf'AeAf = U_{c,cAf}(f'A)$ (note $ff' = 0$) $\subset B$ because $c \in B$, $eAf \subset B$, and B is quadratic. Thus $b \in B$. If $b = bdb$ by regularity, we may assume $d = f'd$. Then $g = db = f'dbf'$ is a nonzero idempotent orthogonal to f, $f+g > f$. On the other hand, B contains $eA(f+g) = eAf + eAg$ since $eAg \subset eAb = eAfAb$ (since A is simple and $f \neq 0$) $= (eAf)(fA)b + b(fA)(eAf)$ (note $bf = 0$) $= U_{eAf,b}(fA) \subset B$. This contraducts the maximality of f, so no $b \neq 0$ exists and $Bf' = 0$. ☐

6.8 Corollary. In a semisimple Artinian algebra A every quadratic ideal B satisfies $BAB \subset B$ (in particular B is strict, even a subalgebra) and it is the intersection of a left and a right ideal: $B = AB \cap BA$. ☐

The d.c.c. on quadratic ideals plays the role for alternative algebras that the d.c.c. on one-sided ideals plays for associative algebras. It is important that these coincide in the semisimple case:

6.9 Corollary. A semisimple associative algebra has d.c.c. on quadratic ideals iff it is Artinian.

Proof. The d.c.c. on quadratic ideals certainly implies the d.c.c. for one-sided ideals. Conversely, if $B_1 \supset B_2 \supset \ldots$

is a chain of quadratic ideals then $AB_1 \supset AB_2 \supset \ldots$ and $B_1A \supset B_2A \supset \ldots$ are chains of one-sided ideals; if these break off, $AB_n = AB_{n+1} = \ldots$ and $B_nA = B_{n+1}A = \ldots$, then so does the original, $B_n = B_{n+1} = \ldots$ since we can recover the quadratic ideals $B_i = AB_i \cap B_iA$ from the one-sided ideals. $\square$

The Inverse Theorem 4.2 (v) - (vi) shows that an element x of a unital algebra is invertible iff the principal quadratic ideal U_xA it generates is all of A. This leads to the useful observation that <u>a proper quadratic ideal cannot contain an invertible element</u>, for as soon as B contains an invertible x it contains all of $U_xA = A$.

An alternative algebra is a division algebra if it is unital and all nonzero elements are invertible. In the associative case this is equivalent to the condition that A have no proper left ideals (or right ideals), but this is not true in the alternative case - we will see that a split Cayley algebra has no proper one-sided ideals, yet is not a division algebra. To be a division algebra there must be no proper quadratic ideals.

6.1α (Division Criterion) A unital alternative algebra is a division algebra iff it has no proper left principal quadratic ideals.

Proof. We have already noted that a division algebra contains no proper quadratic ideals whatsoever, since every nonzero B contains an invertible $x \neq 0$ and hence $B = A$.

Conversely, suppose A has no proper left principal quadratic ideals. By hypothesis each left principal quadratic ideal xA is 0 or A. If $xA = 0$ then $x = x \cdot 1 = 0$, so for $x \neq 0$ we have $xA = A$ and L_x is surjective. By the All-L-Test 4.6, A is a division algebra. ∎

This gives another indication that quadratic ideals should play the role in alternative algebras that one-sided ideals do in associative algebras.

In general, the principal quadratic ideal $U_x A$ generated by x need not contain x. If it does, we say x is <u>regular</u>, so x is regular iff $x = U_x y$ for some y. All invertible elements are regular, but at the other extreme 0 is also regular. In the non-regular case, to find the smallest quadratic ideal containing x we form the <u>weakly principal quadratic ideal</u>

$$B = \Phi x + U_x A .$$

Clearly any quadratic ideal containing x contains both Φx and $U_x A$.

Conversely, the subspace $\Phi x + U_x A$ already constitutes a quadratic ideal which clearly contains x, because

$$U_B\, A \subset U_x\, A\ .$$

Indeed, any element $b = \alpha x + xax$ in B has U-operator $U_b = \alpha^2 U_x + \alpha U_{x,xax} + U_{xax} = U_x\{\alpha^2 I + \alpha(R_a R_x + L_{ax}) + U_a U_x\}$ since $U_{xax} = U_x U_a U_x$ (middle fundamental) and $U_{x,xax} y = x\{y(xax)\} + (xax)\{yx\} = x\{(\{(yx)\}a)x\} + x\{(ax)y\}x$ (right and middle Moufang) $= U_x R_a R_x y + U_x L_{ax} y$.

At the opposite extreme from regular elements are the trivial elements, where z is trivial if $U_z A = 0$. In this case the principal quadratic ideal generated by z vanishes, and the weakly principal quadratic ideal reduces to Φz:

$$\Phi z \text{ is a quadratic ideal if } z \text{ is trivial.}$$

We say z is strictly trivial if it is trivial, $U_z A = 0$, and in addition $z^2 = 0$ (this is automatic in the unital case, $z^2 = U_z 1 = 0$, but not in general). z is strictly trivial in A iff it remains trivial in the unital hull $\hat{A}$, since $U_z \hat{A} = \Phi U_z 1 + U_z A = \Phi z^2 + U_z A$. An algebra contains trivia iff it contains strict trivia: if $z \neq 0$ is trivial, either z is already strictly trivial or else $z^2 \neq 0$ is strictly trivial (note $(z^2)^2 = U_z z^2 = 0$ and $U_{z^2} A = U_z U_z A = 0$).

Trivial elements are bad, and their absence is good. We call an algebra strongly semiprime if it contains no trivial elements (except zero, of course).

An ideal B is trivial if it is trivial as an algebra, $B^2 = 0$. An algebra is semiprime if it has no nonzero trivial ideals. (We will return to this when we discuss radicals.) In the associative case, having trivial elements was equivalent to having trivial ideals. In the alternative case it is easy to see that if B is trivial then all its elements $z \in B$ are (strictly) trivial ($U_z\hat{A} = z(\hat{A}z) \subset BB = 0$). We can rephrase this as

(6.11) Strongly semiprime $\Rightarrow$ semiprime.

The converse of this result presents one of the annoying gaps in the alternative theory. A nontrivial result of Kleinfeld (see Section VI. 4) says that if A has a trivial element $z \neq 0$ then in characteristic $\neq 3$ situations there will be some trivial ideal $B \neq 0$, but the characteristic 3 case is still open, and consequently many structure theorems go through only in characteristic $\neq 3$.

Even in characteristic $\neq 3$, it need not be the case that the ideal B generated by z is itself trivial: although we have $(za)(bz) = 0$ by Middle Moufang, there is no reason why $(za)(zb) \in B^2$ should vanish. Thus the lack of associativity is definitely a complicating factor.

Exercise

1. Show $L(S,C) = \{x \in A \mid xS \subset C\}$ is a quadratic ideal if S is any set and C a left ideal; similarly $R(S,C) = \{x \in A \mid Sx \subset C\}$ if C is a right ideal.
2. If T is a linear operator on A for which there exists another linear operator S with $U_{Tx} = TU_xS$ for all x, show the image $T(B)$ of B under T is a quadratic ideal whenever B is. Particular cases are $T = L_a, R_a, U_a$.
3. Prove that if A has no proper (two-sided) principal quadratic ideals and no trivial elements then A is a division algebra.
4. Show that if A has no proper quadratic ideals it is either a unital division algebra or $U_AA = 0$, and in the latter case show $A = \Phi z$ where $z^2 = 0$ and $\Omega = \Phi/K$ is a field ($K = \{\alpha \mid \alpha z = 0\}$).
5. If x is regular show there are idempotents e, f with $ex = x = xf$.
6. If Φ is a field and $x,y \in A$ satisfy $U_xA \subset \Phi x$, $U_yA \subset \Phi y$ show there is $\alpha \in \Phi$ with $U_xy = \alpha x$, $U_yx = \alpha y$.

7. Show that if x or y is regular in an associative algebra then $xA \cap Ay = xAy$.

8. If A is simple Artinian show any quadratic ideal has the form $B = eAf$ by choosing a maximal quadratic ideal $e \cdot_u A \cdot_u e$ contained in B for a idempotent in an isotope $A^{(u)}$.